

UNIQUENESS OF GRIM HYPERPLANES FOR MEAN CURVATURE FLOWS

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ABSTRACT. In this paper we show that an immersed nontrivial translating soliton for mean curvature flow in \mathbb{R}^{n+1} ($n = 2, 3$) is a grim hyperplane if and only if it is mean convex and has weighted total extrinsic curvature of at most quadratic growth. For an embedded translating soliton Σ with nonnegative scalar curvature, we prove that if the mean curvature of Σ does not change signs on each end, then Σ must have positive scalar curvature unless it is either a hyperplane or a grim hyperplane.

1. INTRODUCTION

A mean curvature flow (MCF) in \mathbb{R}^{n+1} is the negative gradient flow of the volume functional, which can be analyzed from the perspective of partial differential equations as shown by Huisken in [4]. MCF is smooth in a short time and singularities must happen over a longer time. According to the rate of blow-up of the second fundamental form $A(t, p)$ of the hypersurface Σ_t , this finite time singularity T is called type-I, if there exists a constant C_0 such that

$$\sup_{p \in \Sigma_t} |A(t, p)|^2 \leq \frac{C_0}{(T - t)}$$

for all $t < T$. Otherwise this finite time singularity is called type-II.

We will deal with translating solitons which are important in study of type-II singularities.

A complete connected isometrically immersed hypersurface (Σ, Φ) in \mathbb{R}^{n+1} is called a *translating soliton* if its mean curvature vector satisfies

$$\vec{H} = w^\perp,$$

where $w \in \mathbb{R}^{n+1}$ is a unitary vector and w^\perp stands for the orthogonal projection of w onto the normal bundle of Φ . Let ν denote the unit normal along Φ , then it is equivalent to

$$H = -\langle \nu, w \rangle.$$

In particular, considering $f : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$ defined by $f(x) = -\langle x, w \rangle$, then $\bar{\nabla} f = -w$ and $H = \langle \bar{\nabla} f, \nu \rangle$, therefore by definition translating solitons are f -minimal hypersurfaces. Since MCF is invariant under isometries,

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without loss of generality we may suppose that $w = (0, \dots, 0, 1)$, then the function f is defined by $f(x) = -x_{n+1}$ and the L_f -stability operator of Σ is given by

$$(1) \quad L_f = \Delta_f + |A|^2$$

There are some examples of translating solitons: vertical hyperplanes, grim hyperplanes, translating bowl solitons and translating catenoids. In this article we will give a characterization of grim hyperplanes in dimensions 2 and 3.

Recall that a *grim hyperplane* in \mathbb{R}^{n+1} is a hypersurface \mathcal{G} of \mathbb{R}^{n+1} which can be represented parametrically via the embedding $\Phi : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \times \mathbb{R}^{n-1} \rightarrow \mathbb{R}^{n+1}$ defined by

$$\Phi(t, y_1, \dots, y_{n-1}) = (t, y_1, \dots, y_{n-1}, -\ln(\cos t)).$$

The grim hyperplane \mathcal{G} satisfies the translating soliton equation with $w = (0, \dots, 0, 1)$ i.e. it is f -minimal for $f(x_1, \dots, x_{n+1}) = -x_{n+1}$. Also it has positive mean curvature. When $n = 2$ or 3 , there exists a constant $C > 0$ such that

$$(2) \quad \int_{B_R} |A|^2 e^{-f} \leq CR^2$$

for all R sufficiently large. The aim of this article is to prove that indeed the grim hyperplanes are the only ones with these properties when $n = 2, 3$.

Theorem 1. *Let $\Phi : \Sigma^n \rightarrow \mathbb{R}^{n+1}$ be a translating soliton, with $n = 2$ or 3 , which is not a hyperplane. Then Σ is a grim hyperplane if and only if $H = -\langle w, \nu \rangle \geq 0$ and there exists $C > 0$ such that*

$$(3) \quad \int_{B_R} |A|^2 e^{-f} \leq CR^2,$$

for all R sufficiently large, where B_R is the geodesic ball of radius R and $f(x) = -\langle x, w \rangle$.

The expression (3) is not satisfied for $n \geq 4$ (see Proposition 1), thus Theorem 1 is sharp in this sense.

It has been known that if $H \geq 0$ on a translating soliton Σ , then either $H \equiv 0$ on Σ and Σ is a hyperplane, or $H > 0$ everywhere on Σ . Note that both hyperplane and grim hyperplane has vanishing scalar curvature. In [6], Martín-Savas-Halilaj-Smoczyk proved that flat hyperplane and grim hyperplane are the only translating soliton with vanishing scalar curvature. It would be interesting to ask if the following is true.

Problem: Let Σ be a translating soliton with nonnegative scalar curvature S . Is it true that either $S \equiv 0$ on Σ and Σ is a hyperplane or grim hyperplane, or $S > 0$ everywhere on Σ ?

This problem is related to a result proved by Huang-Wu in [3]. Let M be a closed embedded n -dimensional hypersurface in \mathbb{R}^{n+1} with nonnegative scalar curvature. Let M_t be a solution to the mean curvature flow with

initial hypersurface M . Then the scalar curvature of M_t is strictly positive for all $t > 0$.

For complete embedded translating solitons, we have

Theorem 2. *Let (Σ^n, g) be an embedded translating soliton with nonnegative scalar curvature S . Assume H does not change signs on each end. Then either Σ is a hyperplane or a grim hypersurface; or Σ has positive scalar curvature.*

2. TOTAL WEIGHTED EXTRINSIC CURVATURE

In this section we will give the asymptotic properties of the total weighted extrinsic curvatures of grim hyperplanes. We have

$$\partial_t = \sec(t) (\cos t, 0, \dots, 0, \sin t).$$

We choose the unit normal ν to \mathcal{G} to be $\nu = (\sin t, 0, \dots, 0, -\cos t)$. A little computation shows that $\bar{\nabla}_{\partial_t} \nu = (\cos t) \partial_t$ and $\bar{\nabla}_{\partial_{y_i}} \nu = 0$ ($1 \leq i \leq n-1$).

Then the principal curvatures are $\lambda_1 = \cos t$, $\lambda_2 = \dots = \lambda_n = 0$, thus on the coordinates t, y_1, \dots, y_{n-1} the mean curvature only depends on t and is given by $H(t) = \cos t$. Since $t \in (-\frac{\pi}{2}, \frac{\pi}{2})$, we have the norm of the second fundamental form is given by

$$(4) \quad |A|(t) = \cos t = H(t).$$

Now, consider the function $f : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$ defined by $f(x) = -x_{n+1}$, then

$$\langle \bar{\nabla} f, \nu \rangle = \cos t = H.$$

Proposition 1. *The Grim Hyperplane \mathcal{G} in \mathbb{R}^{n+1} satisfies*

$$\lim_{R \rightarrow +\infty} \frac{1}{R^{n-1}} \int_{B_R} |A|^2 e^{-f} = |B^{n-1}(1)| \pi,$$

where B_R is the geodesic ball with center at 0 and radius R and $B^{n-1}(1)$ is the open ball in \mathbb{R}^{n-1} of radius 1 and center at the origin.

Proof of Proposition 1. Observe that f and the metric on \mathcal{G} in the coordinates t, y_1, \dots, y_{n-1} are given by

$$f(t) = \ln(\cos t)$$

and

$$g = \sec^2(t) dt^2 + dy_1^2 + \dots + dy_{n-1}^2.$$

Thus

$$r = \int_0^t \sec(\xi) d\xi = -\ln \left(\tan \left(\frac{1}{2} \left(\frac{\pi}{2} - t \right) \right) \right),$$

we have $t = \frac{\pi}{2} - \eta(r)$, where $\eta(r) = 2 \arctan(e^{-r})$. Then

$$g = dr^2 + dy_1^2 + \dots + dy_{n-1}^2.$$

Besides that $|A|$ and f in the coordinates r, y_1, \dots, y_{n-1} are given by

$$|A|(r) = \sin(\eta(r)),$$

and

$$f(r) = \ln(\sin(\eta(r))).$$

Denoting by $\|\cdot\|$ the standard norm of \mathbb{R}^{n-1} , we have

$$\begin{aligned} B_R &= \left\{ (r, y) \in \mathbb{R} \times \mathbb{R}^{n-1} : r^2 + \|y\|^2 \leq R^2 \right\} \\ &= \left\{ (r, y) \in \mathbb{R} \times \mathbb{R}^{n-1} : -\sqrt{R^2 - \|y\|^2} \leq r \leq \sqrt{R^2 - \|y\|^2}, \quad \|y\| \leq R \right\}. \end{aligned}$$

Since $-\eta'(r) = \sin(\eta(r))$ is an even function, then

$$\begin{aligned} \int_{B_R} |A|^2 e^{-f} &= \int_{\{\|y\| \leq R\}} \left[\int_{-\sqrt{R^2 - \|y\|^2}}^{\sqrt{R^2 - \|y\|^2}} \sin(\eta(r)) dr \right] dy \\ &= \int_{\{\|y\| \leq R\}} \left[\pi - 2\eta\left(\sqrt{R^2 - \|y\|^2}\right) \right] dy \\ &= \pi \int_{\{\|y\| \leq R\}} 1 dy - 2 \int_{\{\|y\| \leq R\}} \eta\left(\sqrt{R^2 - \|y\|^2}\right) dy \\ &= \pi |B^{n-1}(1)| R^{n-1} - 2 \int_0^R \left(\int_{\mathbb{S}_\rho^{n-2}} \eta\left(\sqrt{R^2 - \rho^2}\right) dA \right) d\rho \\ &= \pi |B^{n-1}(1)| R^{n-1} - 2 \text{area}(\mathbb{S}^{n-2}) \int_0^R \eta\left(\sqrt{R^2 - \rho^2}\right) \rho^{n-2} d\rho. \end{aligned}$$

where we have used the co-area formula. Now, letting $\rho = R \sin \theta$ and using the fact $\text{area}(\mathbb{S}^{n-2}) = (n-1) |B^{n-1}(1)|$, we have

$$(5) \quad \frac{1}{R^{n-1}} \int_{B_R} |A|^2 e^{-f} = |B^{n-1}(1)| [\pi - 2(n-1) F_{n-1}(R)],$$

where

$$(6) \quad F_{n-1}(R) = \int_0^{\pi/2} \eta(R \cos \theta) \sin^{n-2} \theta \cos \theta d\theta.$$

Observe that

$$\lim_{R \rightarrow +\infty} \eta(R \cos \theta) \sin^{n-2} \theta \cos \theta = 0 \quad \text{for all } \theta \in \left[0, \frac{\pi}{2}\right].$$

Fixing $R > 0$, we have $|\eta(R \cos \theta) \sin^{n-2} \theta \cos \theta| \leq \frac{\pi}{2} \sin^{n-2} \theta \cos \theta$ for all $\theta \in [0, \pi/2]$. Besides that

$$\int_0^{\pi/2} \sin^{n-2} \theta \cos \theta d\theta = 1/(n-1).$$

Then $\lim_{R \rightarrow +\infty} F_{n-1}(R) = 0$, and hence by (5), we get

$$\lim_{R \rightarrow +\infty} \frac{1}{R^{n-1}} \int_{B_R} |A|^2 e^{-f} = |B^{n-1}(1)| \pi.$$

3. PROOFS OF THEOREM 1 AND THEOREM 2

We begin this section with the following lemma which is in a form more general than we need. The lemma may have its independent interest.

Lemma 1. *Assume that on a complete weighted manifold $(M, \langle \cdot, \cdot \rangle, e^{-f} d\text{vol})$, the functions $u, v \in C^2(M)$, with $u > 0$ and $v \geq 0$ on M , satisfy*

$$(7) \quad \Delta_f u + q(x)u \leq 0 \quad \text{and} \quad \Delta_f v + q(x)v \geq 0,$$

where $q(x) \in C^0(M)$. Suppose that there exists a positive function $\kappa > 0$ on \mathbb{R}^+ satisfying $\frac{t}{\kappa(t)}$ is nonincreasing and

$$\int^{+\infty} \frac{t}{\kappa(t)} dt = +\infty,$$

such that

$$(8) \quad \int_{B_R} v^2 e^{-f} \leq \kappa(R)$$

for all R . Then there exists a constant C such that $v = Cu$.

Remark 1. Without loss of generality, we can assume $\kappa(t) \geq C(1+t^2)$. Some examples of $\kappa(t)$ are Ct^2 , $Ct^2 \log(1+t)$, $Ct^2 \log(1+t) \log \log(3+t)$, \dots .

Proof of Lemma 1. Set $w = \frac{v}{u}$, then $v = wu$, thus by (7) we get

$$\begin{aligned} \Delta_f v &= w \Delta_f u + 2 \langle \nabla w, \nabla u \rangle + u \Delta_f w \\ &\leq -w(qu) + 2 \langle \nabla w, \nabla u \rangle + u \Delta_f w \\ &= -qv + 2 \langle \nabla w, \nabla u \rangle + u \Delta_f w. \end{aligned}$$

Then

$$(9) \quad \Delta_f w \geq -2 \langle \nabla w, \nabla (\ln u) \rangle.$$

On the other hand, let $\varphi \in C_c^\infty(M)$, then by (9), we have

$$\begin{aligned} \int_M \varphi^2 |\nabla w|^2 e^{-f} &= \int_M \langle \varphi^2 \nabla w, \nabla w \rangle e^{-f} \\ &= \int_M \langle \nabla (\varphi^2 w), \nabla w \rangle e^{-f} - 2 \int_M \varphi w \langle \nabla \varphi, \nabla w \rangle e^{-f} \\ &= - \int_M \varphi^2 w (\Delta_f w) e^{-f} - 2 \int_M \varphi w \langle \nabla \varphi, \nabla w \rangle e^{-f} \\ &\leq 2 \int_M \varphi^2 w \langle \nabla w, \nabla (\ln u) \rangle e^{-f} - 2 \int_M \varphi w \langle \nabla \varphi, \nabla w \rangle e^{-f} \\ &= 2 \int_M \langle \varphi \nabla w, w (\varphi \nabla (\ln u) - \nabla \varphi) \rangle \\ &\leq \frac{1}{2} \int_M \varphi^2 |\nabla w|^2 e^{-f} + 2 \int_M w^2 |\varphi \nabla (\ln u) - \nabla \varphi|^2 e^{-f}. \end{aligned}$$

Then

$$(10) \quad \int_M \varphi^2 |\nabla w|^2 e^{-f} \leq 4 \int_M w^2 |\varphi \nabla (\ln u) - \nabla \varphi|^2 e^{-f} \quad \forall \varphi \in C_o^2(M).$$

If $\psi \in C_o^\infty(M)$, then $\varphi = \psi u \in C_o^2(M)$. Besides that, a little computation shows

$$\varphi \nabla (\ln u) - \nabla \varphi = -(\nabla \psi) u,$$

Thus, from (10), we have

$$(11) \quad \begin{aligned} \int_M \psi^2 u^2 |\nabla w|^2 e^{-f} &\leq 4 \int_M w^2 |\nabla \psi|^2 u^2 e^{-f} \\ &= 4 \int_M |\nabla \psi|^2 v^2 e^{-f} \quad \forall \psi \in C_o^\infty(M). \end{aligned}$$

Define functions β, ξ on $[0, +\infty)$ as

$$\beta(t) := \int_0^t \frac{\tau}{\kappa(\tau)} d\tau,$$

and ξ is the inverse function of β . From the hypothesis we know β' is nonincreasing and ξ' is nondecreasing functions on $[0, +\infty)$. Now, we now choose a cutoff function

$$\psi_R(x) = \begin{cases} 1, & \text{on } B_{\xi(R)}; \\ 2 - \frac{\beta(r(x))}{R}, & \text{on } B_{\xi(2R)} \setminus B_{\xi(R)}; \\ 0, & \text{on } M \setminus B_{\xi(2R)}. \end{cases}$$

where $r(x) = d(x, p)$, $p \in M$ is a fixed point and B_R is the geodesic ball with radius R and center p . We see that $|\nabla \psi_R| = \frac{\beta'(r)}{R} = \frac{r}{R\kappa(r)}$. Then, by (8), we get

$$\begin{aligned} \int_{B_{\xi(R)}} u^2 |\nabla w|^2 e^{-f} &= \int_{B_{\xi(R)}} \psi_R^2 u^2 |\nabla w|^2 e^{-f} \\ &\leq \int_M \psi_R^2 u^2 |\nabla w|^2 e^{-f} \\ &\leq 4 \int_M v^2 |\nabla \psi_R|^2 e^{-f} \\ &= 4 \int_{B_{\xi(2R)} \setminus B_{\xi(R)}} v^2 |\nabla \psi_R|^2 e^{-f} \\ &= \frac{4}{R^2} \int_{\xi(R)}^{\xi(2R)} (\beta'(s))^2 \int_{\partial B_s} v^2 e^{-f} dAds. \end{aligned}$$

Here we have used co-area formula. For convenience, we write $V(s) = \int_{B_s} v^2 e^{-f} dV$. Therefore

$$V(s) = \int_0^s \int_{\partial B_\tau} v^2 e^{-f} dAd\tau \leq \kappa(s),$$

and

$$\begin{aligned}
\int_{B_{\xi(R)}} u^2 |\nabla w|^2 e^{-f} &\leq \frac{4}{R^2} \int_{\xi(R)}^{\xi(2R)} (\beta'(s))^2 V'(s) ds \\
&= \frac{4}{R^2} \left[V(s) (\beta'(s))^2 \Big|_{\xi(R)}^{\xi(2R)} - \int_{\xi(R)}^{\xi(2R)} 2V(s) (\beta'(s)) d\beta'(s) \right] \\
&\leq \frac{4}{R^2} \left[V(s) (\beta'(s))^2 \Big|_{\xi(R)}^{\xi(2R)} - 2 \int_{\xi(R)}^{\xi(2R)} s d\beta'(s) \right] \\
&\leq \frac{4}{R^2} \left[V(s) (\beta'(s))^2 \Big|_{\xi(R)}^{\xi(2R)} - 2s\beta'(s) \Big|_{\xi(R)}^{\xi(2R)} + 2 \int_{\xi(R)}^{\xi(2R)} \beta'(s) ds \right] \\
&\leq \frac{4}{R^2} \left[V(s) (\beta'(s))^2 \Big|_{\xi(R)}^{\xi(2R)} - 2s\beta'(s) \Big|_{\xi(R)}^{\xi(2R)} + \beta(s) \Big|_{\xi(R)}^{\xi(2R)} \right] \\
&\leq \frac{4}{R^2} \left[V(s) (\beta'(s))^2 \Big|_{\xi(R)}^{\xi(2R)} - 2s\beta'(s) \Big|_{\xi(R)}^{\xi(2R)} + R \right]
\end{aligned}$$

Since

$$V(s) (\beta'(s))^2 = V(s) \beta'(s) \beta'(s) \leq s \beta'(s),$$

and $\beta'(s) = \frac{s}{\kappa(s)}$, thus Remark 1 implies these terms are bounded, hence when $R \rightarrow +\infty$, all the terms on the right hand side go to zero. So we get

$$\int_M u^2 |\nabla w|^2 e^{-f} = 0.$$

Then $\nabla w \equiv 0$, thus there is a constant C such that $w \equiv C$ and hence $v = Cu$. \square

Definition 1. A two-sided translating soliton Σ is said to be stable if

$$\int_{\Sigma} \left[|\nabla \varphi|^2 - |A|^2 \varphi^2 \right] e^{-f} d\sigma \geq 0 \quad \text{for all } \varphi \in C_0^\infty(\Sigma).$$

As a consequence of Lemma 1, we have the following:

Corollary 1. Let $\Phi : \Sigma^n \rightarrow \mathbb{R}^{n+1}$ be a stable translating soliton and let $\omega \in C^2(\Sigma)$ be a positive solution of the stability equation

$$(12) \quad \Delta_f \omega + |A|^2 \omega = 0.$$

Moreover, if $H \geq 0$ and there exists a constant $C > 0$ such that

$$(13) \quad \int_{B_R} H^2 e^{-f} \leq CR^2 \quad \text{for all } R \text{ large enough.}$$

Then there exists a constant \tilde{C} such that $H = \tilde{C}\omega$. In particular, if $H \not\equiv 0$, then $\tilde{C} \in \mathbb{R} \setminus \{0\}$ and $H > 0$.

Now, we include here a result due to Li and Wang ([5]) which will be needed in the proof our main theorem.

Lemma 2. *Suppose Σ is complete and there exists a nonnegative function $\varphi : \Sigma \rightarrow \mathbb{R}$, not identically zero, such that $(\Delta_f + q)(\varphi) \leq 0$. Then $\Delta_f + q$ is stable.*

Proof. Let Ω be a compact subdomain in Σ and let u be the first eigenfunction satisfying

$$(14) \quad \begin{cases} (\Delta_f + q)u = -\lambda_1(\Omega)u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

We may assume that $u \geq 0$ on Ω . From regularity of u and Hopf Lemma, we have

- $u > 0$ in the interior of Ω .
- $\frac{\partial u}{\partial \nu} < 0$ on $\partial\Omega$, where ν is the outward unit normal of $\partial\Omega$.

Thus, integration by parts on u and φ and also the hypothesis, we have

$$(15) \quad \begin{aligned} \int_{\Omega} u(\Delta_f \varphi) e^{-f} - \int_{\Omega} \varphi(\Delta_f u) e^{-f} &= \int_{\partial\Omega} u \frac{\partial \varphi}{\partial \nu} e^{-f} - \int_{\partial\Omega} \varphi \frac{\partial u}{\partial \nu} e^{-f} \\ &= - \int_{\partial\Omega} \varphi \frac{\partial u}{\partial \nu} e^{-f} \geq 0. \end{aligned}$$

From hypothesis and (14), we have

$$(16) \quad \begin{cases} \Delta_f \varphi + Q\varphi \leq 0, \\ \Delta_f u + Qu = -\lambda_1(\Omega)u. \end{cases}$$

Since $u > 0$, multiplying the first inequality of (16) by u and the second equation by $-\varphi$, and finally both by e^{-f} , we have

$$(17) \quad u(\Delta_f \varphi) e^{-f} - \varphi(\Delta_f u) e^{-f} \leq \lambda_1(\Omega)(\varphi u) e^{-f}$$

Since both $u > 0$ and $\varphi \geq 0$ are not identically zero, then combining (17) with (15), we have $\lambda_1(\Omega) \geq 0$ for all compact subdomains of Σ , then $\lambda_1(f, Q) \geq 0$, therefore $\Delta_f + q$ is stable. \square

We are now ready to give the proof of Theorem 1.

Proof of Theorem 1 Since $\Phi : \Sigma^n \rightarrow \mathbb{R}^{n+1}$ is a translating soliton, then the mean curvature H satisfies $\Delta_f H + |A|^2 H = 0$ (see Proposition 3 in [1]). Since $H \geq 0$ and Σ is a non-planar translating soliton, then H is not identically zero, thus by Lemma 2, Σ is stable and hence the weighted version of a result by Fischer-Colbrie and Schoen [2] guarantees there exists a non-constant positive C^2 -function ω on Σ such that

$$(18) \quad \Delta_f \omega + |A|^2 \omega = 0.$$

As $\frac{H^2}{n} \leq |A|^2$ and $|A|$ satisfies (3), then

$$(19) \quad \int_{B_R} H^2 e^{-f} \leq nCR^2.$$

Then, by Corollary 1 and the condition that $H \geq 0$ and not identically zero, there is a constant $C_1 > 0$ such that

$$(20) \quad H = C_1 \omega.$$

In particular $H > 0$ everywhere on Σ . On the other hand, the Simons equation (see [1] or [6]) implies that

$$(21) \quad |A| \left\{ \Delta_f |A| + |A|^2 |A| \right\} = |\nabla A|^2 - |\nabla |A||^2 \geq 0.$$

Since $|A|$ satisfies (3), then by Lemma 1, $\exists C_2 \geq 0$ such that

$$(22) \quad |A| = C_2 \omega.$$

Besides that Σ^n is not a hyperplane, then $|A|$ is not identically zero, thus $C_2 > 0$. Then by (20) and (22) we have $|A|^2 H^{-2} = \text{constant} > 0$. In particular this function attains its local maximum on Σ . Theorem B in [6] says that Σ is a grim hyperplane if and only if the function $|A|^2 H^{-2}$ attains a local maximum. Therefore Σ is a grim hyperplane. \square

We now prove Theorem 2.

Proof of Theorem 2. To prove Theorem 2, we will need a result of Huang-Wu[3]. Denote by M_+ a connected component of $\{p \in M, H \geq 0 \text{ at } p\}$ that contains a point of positive mean curvature. We say that the mean curvature H changes signs through Γ if Γ is a connected component of ∂M_+ and Γ intersects the boundary of a connected component of $M \setminus \partial M_+$. Theorem 2 of Huang-Wu[3] $S \geq 0$, says that if H changes sign along Γ then Γ is unbounded set. Since we have assumed that H does not changes signs at infinity, H has a sign. Hence either

- (1) $H \equiv 0$, or
- (2) $H \geq 0$ but does not vanish at least one point.

In case (1), Σ must be a hyperplane.

In case (2), if there is point $p \in \Sigma$, such that $S(p) = 0$ then $|A|^2 = H^2 - S \leq H^2$ and equality holds at p . Therefore the function $|A|^2 H^2$ is well defined and attains its maximum at p . By Theorem B in [6] it must be a grim hyperplane. \square

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